

NEVILLE'S PRIMITIVE ELLIPTIC FUNCTIONS: THE CASE  $g_3 = 0$ 

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ABSTRACT. The vanishing of the invariant  $g_3$  attached to a lattice  $\Lambda$  singles out a midpoint lattice and yields a square-root of the associated Weierstrass function  $\wp_\Lambda$ .

Neville's *Jacobian Elliptic Functions* [1] is a peerless classic in its field. It is therefore with some reticence that we draw attention to a minor oversight in its presentation of the primitive functions as meromorphic square-roots of the shifted Weierstrass function  $\wp$ .

The oversight occurs on page 50 of [1]: there it is stated that ‘The zeros of  $\wp z$  are simple, and the branches of  $(\wp z)^{\frac{1}{2}}$  can not be separated’. This is not quite correct: if we denote by  $\Lambda$  the period/pole lattice of  $\wp$  then the zeros of  $\wp$  are simple *except* in case the invariant  $g_3(\Lambda)$  is zero. We note that the statement asserting simplicity of the zeros of  $\wp$  is not made in the prequel [2].

In a little more detail, let  $\Lambda \subset \mathbb{C}$  be any lattice; the associated Weierstrass function  $\wp = \wp_\Lambda$  is then defined by

$$\wp(z) = z^{-2} + \sum_{0 \neq \lambda \in \Lambda} \{(z - \lambda)^{-2} - \lambda^{-2}\}$$

and has  $\Lambda$  as both its period lattice and its pole lattice. The zeros of the derivative  $\wp'$  are precisely those  $z \notin \Lambda$  such that  $2z \in \Lambda$  and they make up three congruent lattices. If

$$\Lambda = \{2n_1\omega_1 + 2n_2\omega_2 : n_1, n_2 \in \mathbb{Z}\}$$

and  $\omega_1 + \omega_2 + \omega_3 = 0$  then these three *midpoint lattices* are  $\omega_1 + \Lambda$ ,  $\omega_2 + \Lambda$ ,  $\omega_3 + \Lambda$ ; the values of  $\wp$  at each point of these lattices are denoted by  $e_1, e_2, e_3$  respectively. Among their many properties, these distinct *midpoint constants* satisfy

$$e_1 + e_2 + e_3 = 0$$

and

$$e_1 e_2 e_3 = g_3/4$$

where the invariant  $g_3 = g_3(\Lambda)$  is defined by

$$g_3 = 140 \sum_{0 \neq \lambda \in \Lambda} \lambda^{-6}.$$

It follows at once that if  $g_3 = 0$  then precisely one of the midpoint constants vanishes: say  $0 = e_p = \wp(\omega_p)$ ; as  $\wp'(\omega_p) = 0$  also,  $\omega_p$  is a double zero of the second-order elliptic function  $\wp$ . As its poles are also double, the Weierstrass function  $\wp$  itself has meromorphic square-roots (by the Weierstrass product theorem, for instance). It also follows that if  $g_3 \neq 0$  then none of the midpoint constants vanishes, so that  $\wp$  has simple zeros and no meromorphic square-roots.

Neville shifts  $\wp$  by the midpoint constants and considers the three functions  $\wp - e_p$  as  $p$  runs over  $\{1, 2, 3\}$ . By design, each of these second-order elliptic functions has double zeros on the corresponding midpoint lattice  $\omega_p + \Lambda$  and so has two meromorphic square-roots; Neville (though with an ingenious change of notation, which we recommend) defines the primitive function  $J_p$  to be the meromorphic square-root of  $\wp - e_p$  that satisfies  $zJ_p(z) \rightarrow 1$  as  $z \rightarrow 0$ . Of course, our observation calls for no correction to any of this: it is simply the case that if  $g_3 = 0$  then one of the midpoint constants is actually zero and need not be subtracted; the corresponding primitive function is then naturally preferred.

To take a particularly straightforward example, let  $\omega_1 = 1$  and  $\omega_2 = i$  so that  $\omega_3 = -1 - i$  and

$$\Lambda = \{2m + 2ni : m, n \in \mathbb{Z}\}$$

is the lattice of (even) Gaussian integers; the union  $\frac{1}{2}\Lambda$  of  $\Lambda$  and its three midpoint lattices is the full lattice of Gaussian integers. As multiplication by  $i$  leaves  $\Lambda$  invariant,

$$\sum_{0 \neq \lambda \in \Lambda} \lambda^{-6} = \sum_{0 \neq \lambda \in \Lambda} (i\lambda)^{-6} = i^{-6} \sum_{0 \neq \lambda \in \Lambda} \lambda^{-6} = - \sum_{0 \neq \lambda \in \Lambda} \lambda^{-6}$$

whence

$$g_3(\Lambda) = 0$$

and a similar calculation reveals that

$$\wp(iz) = -\wp(z).$$

It follows that  $e_3 = \wp(\omega_3) = 0$ : indeed,  $\wp(\omega_2) = \wp(i) = -\wp(1) = -\wp(\omega_1)$  so that  $e_1 + e_2 = 0$  while  $e_1 + e_2 + e_3 = 0$  in any case; of course, a direct computation is also possible. For this lattice, the Weierstrass function  $\wp$  has global meromorphic square-roots, namely  $J_3$  and  $-J_3$ . It may be checked that the identity  $\wp(iz) = -\wp(z)$  implies that  $iJ_3(iz) = J_3(z)$ ; it may also be checked that the same symmetry interchanges the other primitive elliptic functions  $J_1$  and  $J_2$  in the sense  $J_2(z) = iJ_1(iz)$  and  $J_1(z) = iJ_2(iz)$ .

To summarize the general situation: if the invariant  $g_3(\Lambda)$  vanishes, then one of the midpoint constants vanishes, naturally singling out the corresponding midpoint lattice along with the corresponding primitive elliptic function, which is a meromorphic square-root of the Weierstrass function  $\wp_\Lambda$  itself; if  $g_3(\Lambda)$  does not vanish, then  $\wp_\Lambda$  lacks meromorphic square-roots. To put a part of this another way, the invariant  $g_3(\Lambda)$  is the obstruction to the existence of a global meromorphic square-root of  $\wp_\Lambda$ .

Incidentally, an obstruction-theoretic significance also attaches to the invariant

$$g_2 = g_2(\Lambda) = 60 \sum_{0 \neq \lambda \in \Lambda} \lambda^{-4}$$

which satisfies

$$e_2e_3 + e_3e_1 + e_1e_2 = -g_2/4.$$

Let  $\zeta_4$  be the fourth-order Eisenstein function defined by

$$\zeta_4(z) = \sum_{\lambda \in \Lambda} (z - \lambda)^{-4}$$

so that

$$\zeta_4 = \frac{1}{6}\wp'' = \wp^2 - \frac{1}{12}g_2.$$

Evidently, if  $g_2$  is zero then  $\zeta_4$  has the functions  $\pm\wp$  as meromorphic square-roots. Assume instead that  $g_2$  is nonzero and write  $c^2 = \frac{1}{12}g_2$ : if  $\zeta_4 = \wp^2 - c^2 = (\wp - c)(\wp + c)$  is a square then its zeros must be double, so that  $c$  and  $-c$  are midpoint constants; as the three midpoint constants have zero sum, they are  $\pm c$  and 0, whence

$$-\frac{1}{4}g_2 = e_2e_3 + e_3e_1 + e_1e_2 = -c^2 = -\frac{1}{12}g_2$$

and therefore  $g_2$  is zero, contrary to assumption. In short,  $\zeta_4$  admits global meromorphic square-roots precisely when the invariant  $g_2$  vanishes.

#### REFERENCES

- [1] E.H. Neville, *Jacobian Elliptic Functions*, Oxford University Press (1944).
- [2] E.H. Neville, *Elliptic Functions: A Primer*, Pergamon Press (1971).

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